Goals of this second lecture:

* give some techniques for computing GW-invs
* define GK quantum $K$-theory.

Computing degree 1 invariants: counting lines

Assumption: $X = G/P$ with $P$ $G$ maximal parabolic

ie $\text{Ric } X = \mathbb{Z} \iff$ classes of curves given by the degree.

AND $P$ corresponds to a long simple root.

Examples: All $Gr(Km)$: $\ldots \rightarrow k$ all roots are long.

$B_n = \ldots \rightarrow \text{eg } OG(k,n+1)$

$C_n = \ldots \rightarrow$ $IG(k,m)$ is not ok.
Assumption: lines are easy to describe.

Thus [Landsberg - Manivel, Strickland]

Let $Y_1 = F(X) = \text{Fano variety of line} = \{ \text{lines in } X \}$

Then $Y_1$ is $G$-homogeneous and projective so $Y_1 = G/\Delta$ for some $\Delta \subset G$ parabolic.

The simple roots associated to $\Delta$ are obtained as the roots adjacent to the root of $P$ in the Dynkin diagram.

**Ex:** (1) $X = \mathbb{P}^{n-1}$ so type $A_{n-1}$; 

\[ \begin{array}{c}
\text{root of } \Delta \\
\text{root of } P
\end{array} \]

so $Y_1 = F(X) = G/\Delta = Gr(2, n) = \text{lines in } \mathbb{P}^{n-1}$ (OK!)

(2) $X = Gr(k, n)$ 

\[ \begin{array}{c}
\text{root of } \Delta \\
\text{root of } P
\end{array} \]

$Y_1 = F(X) = G/\Delta = \mathcal{F}(k-1, k+1; n$).
Geometry: This says that \( \overline{\pi}_{0,1}(x,1) \) and \( \overline{\pi}_{0,0}(x,1) \) are easy to describe:

\[
\overline{\pi}_{0,1}(x,1) \xrightarrow{ev} X = G/P \quad \Rightarrow \quad \overline{\pi}_{0,0}(x,1) = F(x) \quad \Rightarrow \quad \overline{\pi}_{0,0}(x,1) \cong G/P
\]

\[
\text{Ex.} \quad \text{PE}(k_1, k_{1+l}; n) \Rightarrow X = G/k_1 \text{ for } G \text{ general}.
\]

But recall: GW-invariants are enumerative.

\[
\langle \sigma^k, \sigma^l \rangle \overset{\text{GW}}{=} \# \{ \text{lines meeting } g_1x^m, g_2x^n, g_3x^w \}
\]

for \( g_1, g_2, g_3 \in G \) general.
Fact: \( \mathfrak{S}_2 \) lines meeting \( X^m \) are given by \( q_1 p_1^{-1}(X^m) \).

\[ \{ pt \in \mathbb{F}_2 \} = \mathbb{F}_2 \xrightarrow{p_1} X \]

\[ \mathbb{F}_2 \xrightarrow{x} \text{line of } \mathbb{F}_2 \]

But all this is \( G \)-equivariant so \( q_1 p_1^{-1}(X^m) \) is \( G \)-stable.

and \( \mathbb{C} \) mod \( \mathbb{C} \) is a Schubert variety.

\[ \Rightarrow \text{line meeting } X^m = q_1 p_1^{-1}(X^m) = Y^\wedge \]

for some \( \wedge \).

**Corollary:**

\[ \langle \sigma_1 x, \sigma_1 y, \sigma_1 z \rangle_{X, x} = \# \{ \text{line meeting } x \in \sigma_1 x, \sigma_2 x, \sigma_3 x \} \]

\[ = \# \sigma_1 \hat{X} \wedge \sigma_2 \hat{Y} \wedge \sigma_3 \hat{Z} \]

\[ = \deg ([Y^\wedge] U [\hat{Y}^\wedge] U [\hat{Y}^\wedge]) \]

\[ \Rightarrow \text{quantitative principle: } GW \text{-invariants of } \deg 1 \text{ are classical } (\deg 0) \text{ invariants on another} \]

\[ \text{homogeneous space.} \]

\[ \text{No generalizations in higher degree (with some constants!)} \]

\[ \text{work well for } G \times (X \times Y). \]
Quantum $K$-theory:

First, $K$-theory is a generalization of cohomology (and a special case of oriented cohomology and cobordism).

We assume $X$ is smooth.

$K(X) = \text{free } \mathbb{Z} \text{-module gen by } [\mathcal{F}] \text{ for } \mathcal{F} \text{ flat sheaf}$

$[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}''] \text{ for } 0 \to \mathcal{F}' \to \mathcal{F}'' \to 0$.

$X$ smooth $\Rightarrow$ free $\mathbb{Z}$-module gen by $[\mathcal{F}]$ for sheaf (same rel 0).

Also $= \text{Grothendieck group of } \mathbb{D}^b(X)$.

Ex: For $X = G/P$ we have $(X_u)_{u \in \text{Weyl}}$ Schubert $\nu$

$X_u \to \mathcal{O}_{X_u} \to G_u = [\mathcal{O}_{X_u}] \in K(X)$

$K(X) = \bigoplus_{u \in \text{Weyl}} \mathbb{Z} \mathcal{O}_u$ (compare with cohomology).

For $f: X \to Y$ we have $f^\#: K(Y) \to K(X)$ pull-back.

For $f: X \to Y$ proper we have $f_*: K(X) \to K(Y)$.
Furthermore \( k \) is a mapping via

\[ [E] \cdot [g] = [E \circ g] \]

on \( \mathfrak{e}_k \) free shears.

**Example**: For \( Y, Z \in X \) at \( Y \) and \( Z \) intersect transversely, then \([O_Y] \cdot [O_Z] = [O_{X,Y,Z}]\).

**Example**: \( Y = \mathbb{P}^2 \subset \mathbb{C}P^3 \) plane \( Z = \mathbb{Q}_{2} \subset \mathbb{C}P^3 \) quadric.

Then \([O_Y] \cdot [O_Z] = [O_{\text{conic}}] = [O_{\text{line}}] = [O_{X, \text{line}}] \).

But \( [O_{X, \text{line}}] = [O_{\text{line}}] + [O_{\text{line}}] - [O_{pt}] \).

Because of the exact sequence:

\[ 0 \rightarrow [O_{X, \text{line}}] \rightarrow [O_{\text{line}}] \otimes [O_{\text{line}}] \rightarrow [O_{pt}] \rightarrow 0. \]
Quantum K-theory:

\[ \langle O_{u_1} O_{v_1} O_w \rangle_{x,d} = \prod_{i=1}^3 (ev_{1_i} O_{u_i} \cdot ev_{2_i} O_{v_i} \cdot ev_{3_i} O_w) \in k(q) = \mathbb{Z} \]

Also,

\[ \langle O_u O_v \rangle_{x,d} = \prod_{i=1}^3 (ev_{1_i} O_{u_i} \cdot ev_{2_i} O_{v_i}) \]

Recall for all:

\[ x \wedge v = \sum_{d,w} q^d \langle O_{u_1} O_{v_1} (O_w)^* \rangle_{O_w} \]

If we set:

\[ O_u \ast O_v = \sum_{d,w} q^d \langle O_{u_1} O_{v_1} (O_w)^* \rangle_{O_w} \]

This is not associative.

You need to replace this by:

**Def:**

\[ O_u \ast O_v = \sum_{d,w} q^d N_{u,v}^{wd} O_w \]
with:

\[
N_{u_{v_{w_{d}}}} = \sum_{\gamma} \sum_{d_{0}+\ldots+d_{r}=d} (-1)^{r} \langle \phi_{u}, \phi_{v_{d}}, 0_{k_{y}} \rangle_{d_{0}} \prod_{i=1}^{r} \langle \phi_{k_{y}}, \phi_{k_{s}} \rangle_{d_{i}} \langle \phi_{v}, \phi_{w} \rangle_{d_{r}}
\]

\[
\left( \sum_{u_{v_{w_{d}}}} \right)_{u_{v_{w_{d}}}} \iff \left( \langle \phi_{u}, \phi_{v_{d}}, \phi_{w} \rangle_{x_{d}} \right)_{u_{v_{w_{d}}}}
\]

This is to take into account the geometry of \( \Pi_{0,1,3} (x_{d}) \) and especially curves that degenerate and become reducible.

The above sum "counts" curves of the form

\[
\int_{u_{v_{w_{d}}}} G_{d_{0}} d_{1} \ldots d_{r}
\]

\( x_{u} \)
The sum with sign is because the boundary in $\overline{\Sigma}_{0,3}(X, d)$ is a SND.

\[ d_1 \geq d_2 \geq d \]

\[ d_1 + d_2 = d \]